The mixed problem for the Lamé system in two dimensions

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Abstract

We consider the mixed problem for L the Lamé system of elasticity in a bounded Lipschitz domain $\Omega \subset \mathbf{R}^2$. We suppose that the boundary is written as the union of two disjoint sets, $\partial\Omega = D \cup N$. We take traction data from the space $L^p(N)$ and Dirichlet data from a Sobolev space $W^{1,p}(D)$ and look for a solution u of $Lu = 0$ with the given boundary conditions. We give a scale invariant condition on D and find an exponent $p_0 > 1$ so that for $1 < p < p_0$, we have a unique solution of this boundary value problem with the non-tangential maximal function of the gradient of the solution in $L^p(\partial\Omega)$. We also establish the existence of a unique solution when the data is taken from Hardy spaces and Hardy-Sobolev spaces with p in $(p_1, 1]$ for some $p_1 < 1$.

1   Introduction

We consider the L^p-mixed problem for the Lamé system of elasticity in a Lipschitz domain $\Omega \subset \mathbf{R}^2$. Thus we consider the operator $L = \mu\Delta + (\lambda + \mu)\nabla \operatorname{div}$ acting on vector-valued functions. We assume that the Lamé parameters satisfy $\mu > 0$ and $\lambda > -\mu$ so that the operator L is elliptic. (See the discussion after (2.5) for more details.) We assume that we have a decomposition of the

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boundary into two sets $\partial\Omega = D \cup N$ where $D \subset \partial\Omega$ is a non-empty, open, proper subset of $\partial\Omega$ and $N = \partial\Omega \setminus D$. We let $\partial u/\partial\rho$ denote a traction operator at the boundary. Given data f_N on N and f_D on D, we look for a solution u of the boundary value problem

$$\begin{cases} Lu = 0, & \text{in } \Omega \\ u = f_D, & \text{on } D \\ \frac{\partial u}{\partial\rho} = f_N, & \text{on } N \\ (\nabla u)^* \in L^p(\partial\Omega). \end{cases} \qquad (1.1)$$

Here, $(\nabla u)^*$ denotes the non-tangential maximal function of ∇u. The goal of this paper is to give conditions on D and the data f_D and f_N that will allow us to establish the existence and uniqueness of the solution u and show that the solution u depends continuously on the data.

This investigation is a continuation of work that dates back at least to Dahlberg [8] who studied the Dirichlet problem for the Laplacian in a Lipschitz domain by a careful investigation of harmonic measure. Further developments for the Laplacian include Jerison and Kenig [19] who treated the Neumann problem and the regularity problem in $L^2(\partial\Omega)$ and Dahlberg and Kenig [9] who studied the Neumann and regularity problems in $L^p(\partial\Omega)$ for p between 1 and 2. We point out that it has become standard to refer to the Dirichlet problem with data that has one derivative in some $L^p(\partial\Omega)$ space as the regularity problem. In both the regularity and the Neumann problem, the goal is to establish estimates on the non-tangential maximal function of the gradient of the solution. Other relevant developments include the work of Dahlberg, Kenig, and Verchota [11] who studied the Dirichlet, Neumann, and regularity problems for the Lamé system in $L^2(\partial\Omega)$ and work of Dahlberg and Kenig [10] who studied the regularity and Neumann problems for the Lamé system in $L^p(\partial\Omega)$ in dimension 3. Additional results were obtained for the Lamé system by Mendez and M. Mitrea [22] and Mayboroda and M. Mitrea [21] and again their work is limited to low dimensions.

Another strand of this story is the study of the mixed problem in Lipschitz domains. The author Brown and collaborators including Capogna, Lanzani, and Sykes [2, 20, 29, 30] have established well-posedness of the mixed problem for the Laplacian in restricted classes of Lipschitz domains. I. Mitrea and M. Mitrea [23] have extended these results to cover the Poisson problem for the Laplacian in a wide range of function spaces. These methods have been extended to the Lamé system with I. Mitrea [4], the Stokes system with I. Mitrea, M. Mitrea, and Wright [1], and to the Hodge Laplacian by Gol'dshtein, I. Mitrea, and M. Mitrea [18]. The aforementioned works rely on the use of the Rellich identity with a vector field α chose that $\alpha \cdot \nu$ changes signs as we pass from D to N. Here ν is the outer unit normal to $\partial\Omega$. It seems unlikely that this technique can be applied in a general Lipschitz domain. In two recent papers, the authors and Taylor [26, 33] have developed techniques to investigate the mixed problem for the Laplacian in a general Lipschitz domain and for quite general decompositions of the boundary. Another

interesting approach was taken by Venouziou and Verchota [35] who study the mixed problem for the Laplacian in a class of polyhedral domains. The Ph.D. dissertation of Venouziou [34] includes a discussion of mixed problems for the biharmonic equation in a class of polyhedral domains and some examples where these mixed problems are not solvable. Venouziou also gives a nice summary of early work on the mixed problem.

The work reported here extends the methods of Taylor, Ott, and Brown to the Lamé system in two dimensions. A key ingredient of our approach to the mixed problem (1.1) is estimates for the Green function for the mixed problem. Taylor, Ott, and Brown [31, 33] obtain the needed estimates for the Green function for the Laplacian in all dimensions by the method of de Giorgi [12]. (As we are considering constant coefficient operators, solutions are well-behaved in the interior of the domain. The interesting issue is regularity at the boundary.) However, this approach is not available for elliptic systems. The Green function for the mixed problem in two dimensions was studied recently by Taylor, Kim, and Brown [32]. We note that the work of Dahlberg and Kenig [10] provides estimates for the Green function for the Dirichlet problem and the Neumann problem in three dimensions. However, this approach depends on estimates for the L^p-Dirichlet problem for p near 2. It is not clear how to extend this approach to mixed boundary value problems. The investigation of the mixed problem for elliptic systems in higher dimensions remains an interesting open question.

With the estimates for the Green function in hand, we argue as in the work of Dahlberg and Kenig [9] and use the asymptotic behavior of the Green function to study our boundary value problem for $p \leq 1$. This gives existence of solutions when the data comes from atomic Hardy spaces. Finally, we adapt an argument of Shen [28] to establish results in L^p for $p > 1$ and obtain the following theorem which is the main result of this paper.

Theorem 1.2 *Let $\Omega \subset \mathbf{R}^2$ be a Lipschitz domain and let $D \subset \partial\Omega$ be a non-empty open proper subset satisfying the corkscrew condition (2.1) and the conditions (2.2) and (2.3). Let L be the Lamé operator with coefficients satisfying (2.6).*

1) There exists $p_0 > 1$ so that for $1 < p < p_0$, the L^p-mixed problem is well-posed in the sense that if $f_N \in L^p(N)$, $f_D \in W^{1,p}(\partial\Omega)$, the problem (1.1) has a unique solution which satisfies the estimate

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C(\|f_N\|_{L^p(N)} + \|f_D\|_{W^{1,p}(\partial\Omega)}).$$

The boundary values of u and ∇u exist as non-tangential limits.

2) There exists $p_1 < 1$ so that if $p_1 < p \leq 1$, the L^p-mixed problem is well-posed. Thus if $f_N \in H^p(N)$ and f_D lies in the Hardy-Sobolev space $H^{1,p}(\partial\Omega)$, then there exists a unique solution of the L^p-mixed problem which satisfies

$$\|u\|_{H^{1,p}(\partial\Omega)} + \|\frac{\partial u}{\partial\rho}\|_{H^p(\partial\Omega)} + \|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C(\|f_N\|_{H^p(N)} + \|f_D\|_{H^{1,p}(\partial\Omega)}).$$

We point out that we will require that our Dirichlet data have an extension from D to $\partial\Omega$. The sets D that we consider may not be extension domains for Sobolev spaces. As our solutions (at least for $p \geq 1$) will have boundary values in $W^{1,p}(\partial\Omega)$, it is clear that we need to assume that f_D is the restriction to D of a function which lies in $W^{1,p}(\partial\Omega)$.

Section 2 gives the definitions and assumptions needed in the rest of the paper. Section 3 gives a reverse Hölder estimate for the gradient of a weak solution of the mixed problem. In section 4, we give the fundamental estimate for solutions with atomic data. Uniqueness is treated in section 5 and the proof of the uniqueness assertion of Theorem 1.2 is given in Theorem 5.8. The existence of solutions for the Hardy space problem with $p \leq 1$ is given in Theorem 6.1 of section 6 and the existence for $p > 1$ is indicated in Theorem 6.3 which completes the proof of Theorem 1.2. Two appendices summarize information that is probably known, but not available in a convenient format. Appendix A gives several versions of the Sobolev inequalities that are essential to this work and Appendix B provides a treatment of the regularity problem with data from the Hardy-Sobolev space $H^{1,p}(\partial\Omega)$.

2 Definitions

2.1 Domains

We will assume that Ω is a bounded Lipschitz domain. While our final results hold only in two dimensions, many steps of the argument can be carried out in any dimension. When possible we will give our arguments in $\mathbf{R}^n$, $n \geq 2$, in order to highlight that part of the argument that is truly two-dimensional and to lay out the issues that restrict our argument to two dimensions. We assume Ω is a bounded open set in $\mathbf{R}^n$ and for $M > 0$, $r > 0$, and $x \in \partial\Omega$, we define $Z_r(x)$ by $Z_r(x) = \{y : |x' - y'| < r, |x_n - y_n| < (4M + 2)r\}$. We say that $Z_r(x)$ is a *coordinate cylinder for* $\partial\Omega$ if there is a Lipschitz function $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ with Lipschitz constant M so that

$$
\begin{aligned}
\partial\Omega \cap Z_r(x) &= \{y : y_n = \phi(y')\} \cap Z_r(x) \\
\Omega \cap Z_r(x) &= \{y : y_n > \phi(y')\} \cap Z_r(x).
\end{aligned}
$$

The coordinate system $(y', y_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$ is assumed to be a rotation and translation of the standard coordinate system on $\mathbf{R}^n$.

We say that Ω is a *Lipschitz domain* with constant M if for each $x \in \partial\Omega$, we may find a coordinate cylinder for $\partial\Omega$ centered at x, $Z_r(x)$. Since $\partial\Omega$ is compact, we may find r_0 and a finite collection of coordinate cylinders $\{Z_{r_0}(x_i)\}_{i=1}^N$ which cover $\partial\Omega$ and so that for each i, $Z_{101r_0}(x_i)$ is also a coordinate cylinder.

Our estimates near the boundary will be defined using the non-tangential maximal function. To define this object, we fix $\alpha > 0$ and for $x \in \partial\Omega$, we define the *non-tangential approach region with vertex at x* by $\Gamma(x) = \{y \in \Omega : |x - y| < (1 + \alpha)d(y)\}$ where $d(y) = \text{dist}(y, \partial\Omega)$ denotes the distance from y

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to the boundary. For a function v defined in Ω, we define the *non-tangential maximal function* of v by

$$v^*(x) = \sup_{y \in \Gamma(x)} |v(y)|, \qquad x \in \partial\Omega.$$

It is well-known that the L^p norm with respect to surface measure of the non-tangential maximal functions defined using different choices of α are comparable. Thus, we suppress the dependence of Γ and v^* on the parameter α.

We recall a useful tool from the Ph.D. dissertation of G. Verchota [36, Theorem A.1]. Given a Lipschitz domain $\Omega \subset \mathbf{R}^n$, we may find a family of domains $\{\Omega_k\}_{k=1}^\infty$ with $\bar{\Omega}_k \subset \Omega$ with the properties that a) each Ω_k is a C^∞-domain, b) there are bi-Lipschitz transformations $\Lambda_k : \partial\Omega \to \partial\Omega_k$ such that $\Lambda_k(x)$ converges to x. Using Verchota's construction we can see that the $\Lambda_k(x)$ converges to x non-tangentially as $k \to \infty$, i.e. if the constant α used to define the approach regions $\Gamma(x)$ is sufficiently large, then $\Lambda_k(x) \in \Gamma(x)$ for all k.

We define a star-shaped Lipschitz domain. These domains will be useful because they have sufficiently regular geometry that we can establish estimates that are uniform over the class of domains. Let $x \in \mathbf{R}^n$, $r > 0$, $M > 0$, and $\phi : \mathbf{S}^{n-1} \to [1, 1 + M]$. We say that Ω is a *star-shaped Lipschitz domain* with center x, constant M, and scale r if ϕ is Lipschitz with constant M and

$$\Omega = x + \{y : |y| < r\phi(y/|y|)\}.$$

We introduce *boundary balls* (or intervals if $n = 2$). For $x \in \partial\Omega$ and $r \in (0, 100r_0)$, we let $\Delta_r(x) = Z_r(x) \cap \partial\Omega$. If $x \in \Omega$ and $r \in (0, 100r_0)$, we define *local domains* as follows. If $d(x) = \mathrm{dist}(x, \partial\Omega) > r$, then $\Omega_r(x) = B_r(x)$, the ball centered at x with radius r. If $d(x) \leq r$, then x lies in one of the coordinate cylinders $Z_{r_0}(x_i) = Z_i$. If $x = (x', x_n)$ in the coordinate system for the cylinder Z_i, then we put $\hat{x} = (x', \phi(x'))$ and define $\Omega_r(x) = Z_r(\hat{x})$. We let $x^* = (x', \phi(x') + (4M + 1)re_n)$ and observe that the domains $\Omega_r(x)$ are star-shaped with respect to each point in $B_{r/2}(x^*)$, thanks to our choice of M. This allows us to prove scale-invariant Sobolev inequalities and Korn inequalities in these domains. If x is on or near the boundary, then x may lie in several of the coordinate cylinders which cover the boundary and thus we have several choices for $\Delta_r(x)$ and $\Omega_r(x)$. Our estimates will hold for any such choice with the condition that if several of these objects appear in an estimate we make a consistent choice of coordinate cylinder to define them. Note that in earlier works, we have sometimes used $S_r(x) = \partial\Omega \cap B_r(x)$ and $\tilde{S}_r(x) = \Omega \cap B_r(x)$, $x \in \Omega$, in place of $\Delta_r(x)$ and $\Omega_r(x)$. Each set of definitions has its advantages and at one point in the argument below we will find it convenient to use $S_r(x)$ in place of $\Delta_r(x)$.

We specify our decomposition of the boundary by giving our assumptions on D. We assume $D \subset \partial\Omega$ is a non-empty proper open subset of $\partial\Omega$. We will assume that D satisfies a corkscrew condition. To describe this condition, let $\Lambda \subset \partial\Omega$ denote the boundary of D in $\partial\Omega$ and $\delta(x) = \mathrm{dist}(x, \Lambda)$. We say that D satisfies the corkscrew condition if for all $x \in \Lambda$ and r in $(0, 100r_0)$, there exists

$x_r \in D$ so that $|x_r - x| < r$ and $\delta(x_r) > M^{-1}r$. In Taylor, Ott, and Brown [33, Section 2], we show that if D satisfies the corkscrew condition, we have the following consequence:

If $x \in D$ and $r \in (0, 100r_0)$, then there exists $x_r \in D$ so that
$|x - x_r| < r$ and $\Delta_{M^{-1}r}(x_r) \subset D$. $\qquad\qquad$ (2.1)

We note that the extreme cases $D = \emptyset$ or $\partial\Omega$ of the mixed problem (1.1) correspond to the traction and regularity problems, respectively. As these have been studied elsewhere, we exclude these cases from our discussion in this paper.

We also require that D satisfy the following conditions that involve the integrability of δ. To state these conditions, for $r \in (0, 100r_0)$, we let $\delta_r(x) = \min(\delta(x), r)$ and then we require that

$$\int_{\Delta_r(x)} \delta_r^t \, d\sigma \approx r^{n-1+t}, \qquad t > -1 + \epsilon \qquad\qquad (2.2)$$

$$\int_{\Omega_r(x)} \delta_r^t \, dy \approx r^{n+t}, \qquad t > -2 + \epsilon. \qquad\qquad (2.3)$$

In our main results there will be a restriction on ϵ. The condition on ϵ depends on the L^q-index in the reverse Hölder inequality in Theorem 3.1 and we refer the reader to section 4 for more details. For the moment, we observe that given M we will find a positive value of ϵ for which the conditions (2.2) and (2.3) allow us to solve the mixed problem. While the conditions (2.2) (2.3) may seem rather mysterious, they are closely related to the dimension of the set Λ. The work of Taylor, Ott, and Brown [33, Lemmata 2.4, 2.5] shows that the conditions (2.2) and (2.3) follow if we assume that the set Λ is of dimension $n - 2 + \epsilon$. In the setting treated in [26], we have $\epsilon = 0$. As the conditions (2.2) and (2.3) become more restrictive as ϵ decreases, the results of this paper always apply in the domains considered in [26].

Example. We close this sub-section by giving an example of a domain that satisfies our conditions and illustrates that even in two dimensions, the set D can be fairly complicated. Thus let $\Omega = \{x : |x| < 1\}$ be the disk and let D be the set $\cup_{k=1}^{\infty}\{(\cos\theta, \sin\theta) : \pi/2^{2k+1} < \theta < \pi/2^{2k}\}$. It is not difficult to see that this domain will satisfy (2.1) and (2.2) and (2.3) with $\epsilon = 0$.

2.2 Function spaces

We will consider L^p spaces with respect to Lebesgue measure on domains in $\Omega \subset \mathbf{R}^n$ and with respect to surface measure, σ, on the boundary of Ω, $\partial\Omega$. These will be denoted $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively.

We let $W^{1,p}(\Omega)$ denote the standard Sobolev space of functions having one derivative in $L^p(\Omega)$ with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_{\Omega} |\nabla u|^p + r_0^{-p}|u|^p \, dy \right)^{1/p}$$

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at least for $1 \leq p < \infty$. The factor of r_0 guarantees that the two terms in the norm have the same homogeneity when rescaling. Our functions will generally take values in $\mathbf{R}^n$, though we will not indicate the range in our notation.

For $D \subset \partial\Omega$, we define $W_D^{1,2}(\Omega)$ as the closure in $W^{1,2}(\Omega)$ of the functions which are smooth in the closure of Ω and which vanish in a neighborhood of D. We let $W_D^{-1,2}(\Omega)$ denote the dual of $W_D^{1,2}(\Omega)$. As noted all domains in this paper will be Lipschitz and thus we have the trace operator which is a continuous map from $W^{1,2}(\Omega)$ into $L^2(\partial\Omega)$ and extends the operation of restricting a smooth function to the boundary. We let $W_D^{1/2,2}(\partial\Omega)$ denote the image of $W_D^{1,2}(\Omega)$ under the trace map and then $W_D^{-1/2,2}(\partial\Omega)$ denotes the dual of $W_D^{1/2,2}(\partial\Omega)$.

We will also need to consider Sobolev spaces on the boundary. For a function u in $C^\infty(\bar\Omega)$ we define a family of tangential derivatives

$$\frac{\partial u}{\partial \tau_{ij}} = \nu_i \frac{\partial u}{\partial x_j} - \nu_j \frac{\partial u}{\partial x_i}, \qquad i, j = 1, \ldots, n, i \neq j$$

where ν denotes the unit outer normal to the boundary. For a function u which is smooth in $\bar\Omega$, we can define the tangential gradient by

$$\nabla_t u^\alpha = \nabla u^\alpha - \nu \nabla u^\alpha \cdot \nu, \qquad \alpha = 1, \ldots, n.$$

Note that we have

$$|\nabla_t u|^2 \approx \sum_{1 \leq i < j \leq n} |\frac{\partial u}{\partial \tau_{ij}}|^2$$

and for $1 \leq p < \infty$, the Sobolev space $W^{1,p}(\partial\Omega)$ is the closure of the boundary values of smooth functions in the norm

$$\|u\|_{W^{1,p}(\partial\Omega)} = (\|\nabla_t u\|_{L^p(\partial\Omega)}^p + r_0^{-p} \|u\|_{L^p(\partial\Omega)}^p)^{1/p}.$$

Before defining Hardy spaces on the boundary, we recall the definition of the spaces of Hölder continuous functions. For K a compact subset of $\bar\Omega$ and $0 < \alpha \leq 1$, we define the Hölder space $C^\alpha(K)$ to be the collection of functions f on K for which the norm below is finite

$$\|f\|_{C^\alpha(K)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} + r_0^{-\alpha} \sup_K |f(x)|.$$

If Ω is a Lipschitz domain in dimension n and $N \subset \partial\Omega$ is a closed set, we introduce the Hardy space $H^p(N)$ for $(n - 1)/n < p \leq 1$. We say that a is an atom for $\partial\Omega$ if a is supported in a boundary ball $\Delta_r(x)$, a satisfies $\|a\|_{L^\infty(\partial\Omega)} \leq 1/\sigma(\Delta_r(x))$ and $\int_{\Delta_r(x)} a \, d\sigma = 0$. We say that a is an atom for N if $a = \tilde{a}|_N$ for some $\tilde{a}$, an atom for $\partial\Omega$. As we shall see these atoms are building blocks for the spaces $H^p(N)$ which arise naturally in the study of L^p-mixed problem for $p \leq 1$. We note that our atoms are normalized for the space H^1. When we introduce H^p spaces with $p < 1$, we will need to introduce powers of $\sigma(\Delta_r(x))$ into the norm to compensate for the fact that the atoms are not normalized in H^p.

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For p satisfying $(n-1)/n < p \le 1$, we say that f lies in the *Hardy space* $H^p(N)$ if there exists a sequence of atoms for N, $\{a_j\}_{j=1}^\infty$ and coefficients $\{\lambda_j\}_{j=1}^\infty \subset (0,\infty)$ so that $f = \sum_{j=1}^\infty \lambda_j a_j$ and $\sum_{j=1}^\infty \lambda_j^p \sigma(\Delta_{r_j}(x_j))^{1-p} < \infty$ where the atom a_j is supported in a boundary ball $\Delta_{r_j}(x_j)$. We define a quasi-norm for $p \le 1$ by

$$\|f\|_{H^p(N)} = \inf \left(\sum_j \lambda_j^p \sigma(\Delta_{r_j}(x_j))^{1-p} \right)^{1/p}$$

where the infimum is taken over all representations of f as a sum of atoms. When $p = 1$, the sum converges in L^1 and we have $H^1(N) \subset L^1(N)$. However, for $p < 1$ elements of $H^p(N)$ may not be functions. Rather they are defined as linear functionals on a space of smooth functions. It is well-known that elements of $H^p(\partial\Omega)$ give continuous linear functionals on the space $C^\alpha(\partial\Omega)$ where $\alpha = (n-1)(1/p-1)$, for $1 > p > (n-1)/n$. It is a straightforward consequence of our definition that elements of $H^p(N)$ give continuous linear functionals on $C_D^\alpha(\partial\Omega)$, the collection of functions in $C^\alpha(\partial\Omega)$ which vanish on D. We observe that it is an immediate consequence of the definition that an element f from $H^p(N)$ has an extension to $H^p(\partial\Omega)$ with the bound $\|f\|_{H^p(\partial\Omega)} \le C\|f\|_{H^p(N)}$ for any $C > 1$. One consequence of our main theorem is that there is a bounded linear extension operator from $H^p(N)$ to $H^p(\partial\Omega)$. However, it would be interesting to develop a better understanding of these spaces.

We define Hardy-Sobolev spaces $H^{1,p}(D)$ as follows. We say that A is a 1-atom for $\partial\Omega$ if A is supported in a boundary ball $\Delta_r(x)$ and $\|\nabla_t A\|_{L^\infty(\partial\Omega)} \le 1/\sigma(\Delta_r(x))$. We say that A is an atom for D, if A is the restriction to D of an atom for $\partial\Omega$. Finally, if p satisfies $(n-1)/n < p \le 1$, we say u is in $H^{1,p}(D)$ if there is a sequence of atoms $\{A_j\}$ for D with each A_j supported in a boundary ball $\Delta_j = \Delta_{r_j}(x_j)$ and a sequence of non-negative real numbers $\{\lambda_j\}_{j=1}^\infty$ with $\sum_j \lambda_j \sigma(\Delta_j)^{1-p} < \infty$ and $u = \sum_j \lambda_j A_j$. We define a quasi-norm on $H^{1,p}(\partial\Omega)$ as $\|u\|_{H^{1,p}(\partial\Omega)} = \inf(\sum_j \lambda_j^p \sigma(\Delta_j)^{1-p})^{1/p}$ where the infinum is taken over all possible representations of u. It is well-known that one may define atoms using an L^t space $1 < t < \infty$ instead of L^∞. Thus, if we fix t with $1 < t < \infty$ and replace the condition $\|a\|_{L^\infty(\Delta_r(x))} \le \sigma(\Delta_r(x))^{-1}$ by $\|a\|_{L^t(\Delta_r(x))} \le \sigma(\Delta_r(x))^{-1/t'}$, then the resulting Hardy space is independent of t. A similar result holds for the Hardy-Sobolev space. The work of Mitrea and Wright [24, sections 2.2-2.3] provides an exposition of these facts in the setting of the boundary of a Lipschitz domain.

2.3 The weak formulation of the mixed problem

Next we recall the Lamé operator L and give several well-known estimates for solutions of the Lamé system, $Lu = 0$. The operator L will be written as

$$(Lu)^\alpha = \frac{\partial}{\partial x_i} a_{\alpha\beta}^{ij} \frac{\partial}{\partial x_j} u^\beta, \qquad \alpha, \beta, i, j = 1, \ldots, n.$$

Here and throughout this paper, we will use the summation convention and thus we sum on the repeated indices i, j and β in the above expression. We will consider the traction operator at the boundary

$$\frac{\partial u^{\alpha}}{\partial \rho} = \nu_i a^{ij}_{\alpha\beta} \frac{\partial u^{\beta}}{\partial x_j}.$$

The coefficients $a^{ij}_{\alpha\beta}$ we consider are given by

$$a^{ij}_{\alpha\beta} = a^{ij}_{\alpha\beta}(s) = \mu\delta_{ij}\delta_{\alpha\beta} + s\delta_{i\beta}\delta_{j\alpha} + (\lambda + \mu - s)\delta_{i\alpha}\delta_{j\beta} \tag{2.4}$$

where δ_{ij} is the Kronecker symbol, λ and μ are the Lamé parameters and s is an artificial parameter that has been introduced in several different settings for mathematical reasons. Note that

$$\frac{\partial}{\partial x_i}(\delta_{i\beta}\delta_{j\alpha} - \delta_{i\alpha}\delta_{j\beta})\frac{\partial}{\partial x_j} = 0.$$

Thus, the differential operator L does not depend on the value of s. However, the traction operator $\partial u/\partial \rho$ does depend on s.

We will require that our operator satisfy the ellipticity condition

$$a^{ij}_{\alpha\beta}\xi^i_{\alpha}\xi^j_{\beta} \geq M^{-1}|\frac{\xi + \xi^t}{2}|^2, \qquad \xi \in \mathbf{R}^{n\times n}. \tag{2.5}$$

This will hold if s, λ, and μ satisfy

$$s \in [0,\mu], \qquad \mu > 0, \qquad \text{and} \qquad 2\mu + n\lambda > 0. \tag{2.6}$$

We note several consequences of this ellipticity assumption (2.6) which come under the name of Korn inequalities. First we have the global coercivity estimate

$$c\int_{\Omega} |\nabla u|^2\, dy \leq \int_{\Omega} a^{ij}_{\alpha\beta}\frac{\partial u^{\beta}}{\partial y_j}\frac{\partial u^{\alpha}}{\partial y_i}\, dy, \qquad u \in W^{1,2}_D(\Omega). \tag{2.7}$$

This will hold if D is a non-empty open subset of the boundary. The constant will depend on Ω and D and the constant will be bounded away from zero for a family of operators which satisfy (2.5) for a fixed M.

For $u \in W^{1,2}(\Omega)$, any local domain $\Omega_{\rho}(x)$, and any constant vector $c \in \mathbf{R}^n$, we have

$$\int_{\Omega_{\rho}(x)} |\nabla u|^2\, dy \leq C(M)\int_{\Omega_{\rho}(x)} a^{ij}_{\alpha\beta}\frac{\partial u^{\beta}}{\partial y_j}\frac{\partial u^{\alpha}}{\partial y_i} + \rho^{-2}|u - c|^2\, dy. \tag{2.8}$$

The constant depends only on M and the estimate does not depend on the properties of D and, in fact, we do not require that u vanish on D for the estimate (2.8) to hold. A proof of (2.8) may be found in the monograph of Oleĭnik, Shamaev, and Yosifian [25, Theorem 2.10]. Throughout this paper, we will consider the Lamé operator (and its generalizations) with coefficients

given by (2.4) and satisfying (2.6). As a consequence of these assumptions, we obtain the estimates (2.5), (2.7), and (2.8).

We are ready to give the definition of a weak solution of the mixed problem. Given $f \in W_D^{-1,2}(\Omega)$ and $f_N \in W^{-1/2,2}(\partial\Omega)$, we consider the boundary value problem

$$
\begin{cases}
Lu = f, & \text{in } \Omega \\
u = 0, & \text{on } D \\
\frac{\partial u}{\partial \rho} = f_N, & \text{on } N.
\end{cases}
\tag{2.9}
$$

We say that u is a *weak solution of (2.9)* if $u \in W_D^{1,2}(\Omega)$ and

$$
\int_\Omega a_{\alpha\beta}^{ij} \frac{\partial u^\beta}{\partial y_j} \frac{\partial \phi^\alpha}{\partial y_i} \, dy = \langle f_N, \phi \rangle_{\partial\Omega} - \langle f, \phi \rangle_\Omega, \qquad \phi \in W_D^{1,2}(\Omega).
$$

We are using $\langle \cdot, \cdot \rangle_\Omega$ and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ to denote the pairings of duality on $W_D^{-1,2}(\Omega) \times W_D^{1,2}(\Omega)$ and $W_D^{-1/2,2}(\partial\Omega) \times W_D^{1/2,2}(\partial\Omega)$, respectively. As we assume that the coefficients of L satisfy (2.5) and we assume D is a nonempty open set of $\partial\Omega$, we have the coercivity estimate (2.7) and a Poincaré inequality (2.10). If $u \in W_D^{1,p}(\Omega)$, then we have that

$$
\|u\|_{L^p(\Omega)} \leq C r_0 \|\nabla u\|_{L^p(\Omega)}.
\tag{2.10}
$$

Thus the existence and uniqueness of solutions to (2.9) is a straightforward consequence of standard Hilbert space theory. The solution of (2.9) will satisfy

$$
\|\nabla u\|_{W_D^{1,2}(\Omega)} \leq C(\|f\|_{W^{-1,2}(\Omega)} + \|f_N\|_{W_D^{-1/2,2}(\partial\Omega)}).
$$

Now that we have introduced the traction operator, $\partial/\partial\rho$, we describe the sense in which we take boundary values of $\partial u/\partial\rho$ when $p < 1$. As the boundary values may not be a function, it is not enough to ask for non-tangential limits a.e. Thus for $p < 1$, and $f \in H^p(\partial\Omega)$, we say that $\partial u/\partial\rho = f$ on $\partial\Omega$, if we have

$$
\lim_{k\to\infty} \int_{\partial\Omega_k} \phi^\beta (\frac{\partial u}{\partial\rho})^\beta \, d\sigma = \langle f, \phi \rangle_{\partial\Omega}, \qquad \phi \in C^\alpha(\bar{\Omega})
\tag{2.11}
$$

where $\Omega_k \subset \Omega$ is a family of approximating domains as constructed by Verchota. If f_N lies in $H^p(N)$, we say that $\partial u/\partial\rho = f_N$ on N if we have (2.11) for ϕ which lie in $C_D^\alpha(\partial\Omega)$.

One technical point about our argument for the mixed problem is that we do not have solutions in smooth domains as a starting point. A common strategy in studying the Dirichlet and Neumann problems is to approximate a Lipschitz domain by a sequence of smooth domains, prove a uniform estimate in the approximating domains, and take a limit. It is an interesting question to find an approximation of the mixed problems by problems which have smooth solutions.

Finally, we make a note about the constants in our estimates. Many of our estimates are of a local nature and hold on scales r with $0 < r < r_0$ and

with a constant that depends only on the parameter M that appears in the definition of a Lipschitz domain, the corkscrew condition, the constants in the estimates (2.2), and (2.3), and any L^p indices that appear in the estimate. We will say a constant depends on the global character of Ω if the constant also depends on the collection of coordinate cylinders appearing in the definition of the Lipschitz domain and the constant in the Korn inequality (2.7).

3 A reverse Hölder inequality for weak solutions

The first step of our argument is to show that if u is a solution of (2.9) with nice data, then ∇u in $L^q(\Omega)$ for some $q > 2$. We use the reverse Hölder argument of Gehring [14] and Giaquinta and Modica [16]. The argument is similar to one given in our joint work with Taylor [26, 33], but a few changes are needed since the operators involved may not be strongly elliptic. The next Theorem gives a precise formulation of our result. We note that the proof of this result only requires the condition (2.1) on D.

Theorem 3.1 *Let $\Omega \subset \mathbf{R}^n$ be a Lipschitz domain and suppose that D satisfies (2.1) and that L is the Lamé system with coefficients satisfying (2.6). Suppose that u is a solution of (2.9) with $f = 0$ and f_N a function. We may find $q_0 > 2$ so that if $2 < q < q_0$ and $f_N \in L^{q(n-1)/n}(N)$, then for any local domain $\Omega_r(x)$ we have*

$$\left(\fint_{\Omega_r(x)} |\nabla u|^q \, dy \right)^{1/q}$$
$$\leq C \left[\fint_{\Omega_{2r}(x)} |\nabla u| \, dy + \left(\fint_{\partial\Omega \cap \partial\Omega_{2r}(x)} |f_N|^{q(n-1)/n} \, d\sigma \right)^{n/((n-1)q)} \right].$$
$$(3.2)$$

The value of q_0 depends only on M. The constant in this estimate depends only on M, the dimension n, and q.

In the above theorem and throughout this paper, we use $\fint_E f$ to denote the average of f over a set E and we adopt the convention that the average over the empty set is zero.

We introduce the notation $\bar{u}_{x,r}$ which will be useful for the arguments of this section. For u a function defined on Ω, $x \in \Omega$ and $r \in (0, 100r_0)$, we let $\bar{u}_{x,r} = 0$ if $\delta(x) \leq r$ and $\bar{u}_{x,r} = \fint_{\Omega_r(x)} u(y) \, dy$ if $\delta(x) > r$. We observe that one useful feature of this definition is that if η is in $C^\infty(\bar{\Omega})$ and $\eta = 0$ outside $\Omega_r(x)$, and $u \in W_D^{1,2}(\Omega)$, then $\eta(u - \bar{u}_{x,r})$ lies in $W_D^{1,2}(\Omega)$.

We give several formulations of the Sobolev inequalities that we will use in the sequel. The proofs of the estimates (3.3) and (3.4) are given in Appendix A.

These take advantage of the corkscrew condition for D (2.1) and the definition of $\bar{u}_{x,r}$. Let p and q satisfy $1 \leq p < n$ and $1/q = 1/p - 1/n$. Then for $x \in \Omega$ and $0 < \rho < r < 100 r_0$, we have

$$\left(\fint_{\Omega_\rho(x)} |u - \bar{u}_{x,\rho}|^q \, dy \right)^{1/q} \leq \frac{C(M,p,n) r^{n-1}}{(r-\rho)^{n-1}} \left(\int_{\Omega_r(x)} |\nabla u|^p \, dy \right)^{1/p}. \qquad (3.3)$$

For $1 \leq p < n$, we define q by $1/q = n/(p(n-1)) - 1/(n-1)$. We observe that for all local domains $\Omega_\rho(x)$ and $\Omega_r(x)$ with $0 < \rho < r$, we have

$$\left(\fint_{\partial\Omega_\rho(x) \cap \partial\Omega} |u - \bar{u}_{x,\rho}|^q \, d\sigma \right)^{1/q} \leq \frac{C(M,p,n) r^{n-1}}{(r-\rho)^{n-1}} \left(\int_{\Omega_r(x)} |\nabla u|^p \, dy \right)^{1/p}. \qquad (3.4)$$

We turn to the proof of the reverse Hölder inequality in Theorem 3.1. The proof will require the following auxiliary function

$$P_r f(x) = \sup_{0 < s < r} \frac{1}{s^{n-1}} \int_{\partial\Omega \cap \partial\Omega_s(x)} |f| \, d\sigma.$$

When we apply P_r to the traction data f_N we will assume that f_N is extended to be a function on $\partial\Omega$ by setting $f_N(x) = 0$ on D. As observed in [26], for $1 < p \leq \infty$, $q = pn/(n-1)$, we have

$$\|P_r f\|_{L^q(\Omega)} \leq C \|f\|_{L^p(\partial\Omega)}. \qquad (3.5)$$

Proof of Theorem 3.1. We let u be a weak solution of (2.9) with $f = 0$ and f_N a function. We fix $x \in \partial\Omega$, ρ and r with $r/2 < \rho < r < r_0$ and set $\epsilon = (r-\rho)/2$. We claim that with p satisfying $p > 2(n-1)/n$, we have

$$\left(\fint_{\Omega_{r/2}(x)} |\nabla u|^2 \, dy \right)^{1/2}$$

$$\leq C \left[\left(\fint_{\Omega_r(x)} |\nabla u| \, dy \right) + \left(\frac{1}{r^{n-1}} \int_{\partial\Omega_r(x) \cap \partial\Omega} |f_N|^p \, d\sigma \right)^{1/p} \right]. \qquad (3.6)$$

Observe that the last term is bounded by $P_{2r}(|f_N|^p)^{1/p}(x)$. Theorem 3.1 will follow from (3.6) using the argument in Giaquinta [15, p. 122], and the estimate (3.5).

Thus, we turn to the proof of (3.6). We let η be a cutoff function which is one on $\Omega_\rho(x)$ and zero outside $\Omega_{\rho+\epsilon}(x)$ and observe that $v = \eta^2(u - \bar{u}_{x,\rho+\epsilon}) \in W_D^{1,2}(\Omega)$. We let $E = \int_{\Omega_{\rho+\epsilon}(x)} |u||\nabla u| \eta |\nabla \eta| \, dy$. Using the product rule, the local ellipticity assumption (2.8), and that u is a solution, we obtain

$$\int_{\Omega_{\rho+\epsilon}(x)} \eta^2 |\nabla u|^2 \, dy = \int_{\Omega_{\rho+\epsilon}(x)} |\nabla \eta (u - \bar{u}_{x,\rho+\epsilon})|^2 \, dy + CE$$

$$\leq \int_\Omega a_{\alpha\beta}^{ij} \frac{\partial u^\beta}{\partial y_j} \frac{\partial v^\alpha}{\partial y_i} \, dy + CE \qquad (3.7)$$

$$= \int_{\partial\Omega} f_N^\beta v^\beta \, d\sigma + CE.$$

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We apply Hölder's inequality, Young's inequality with ϵ's, and the boundary Poincaré inequality (3.4) and obtain for any $\gamma \in \mathbf{R}$ that

$$
| \int_{\partial \Omega} f_N^\beta v^\beta \, d\sigma | \leq r^\gamma \left(\int_{\partial \Omega \cap \partial \Omega_{\rho+\epsilon}(x)} |f_N|^p \, d\sigma \right)^{2/p}
$$
$$
+ \frac{C r^{-\gamma}}{(1 - \rho/r)^{2n-2}} \left(\int_{\Omega_{\rho+2\epsilon}(x)} |\nabla u|^t \, dy \right)^{2/t},
\tag{3.8}
$$

where $p > 1$ and p and t are related by $1/t = 1/n + (n-1)/(np')$. We return to (3.7) and use (3.8) to estimate the boundary term. To handle the error term E, we use Cauchy's inequality with ϵ's to subtract the term ∇u in E from the left-hand side of (3.7) and conclude that

$$
\int_{\Omega_{\rho+\epsilon}(x)} \eta^2 |\nabla u|^2 \, dy \leq C(r^{-2} \int_{\Omega_{\rho+\epsilon}(x)} |u - \bar{u}_{x,\rho+\epsilon}|^2 \, dy + r^\gamma \left(\int_{\partial \Omega \cap \partial \Omega_{\rho+\epsilon}(x)} |f_N|^p \, d\sigma \right)^{2/p}
$$
$$
+ \frac{r^{-\gamma}}{(1 - \rho/r)^{2n-2}} \left(\int_{\Omega_{\rho+2\epsilon}(x)} |\nabla u|^t \, dy \right)^{2/t}).
\tag{3.9}
$$

We use (3.3) to estimate the first term in (3.9), choose γ so that $n + \gamma = 2n/p$ and hence $n - \gamma = 2(n-1)/p$, divide by r^n, and recall that $2\epsilon = r - \rho$ to obtain

$$
\fint_{\Omega_{\rho+\epsilon}(x)} \eta^2 |\nabla u|^2 \, dy \leq C[\left(\frac{1}{r^{n-1}} \int_{\partial \Omega_r(x) \partial \Omega} |f_N|^p \, d\sigma \right)^{2/p}
$$
$$
+ \frac{1}{(1 - (\rho/r))^{2n-2}} (\left(\fint_{\Omega_r(x)} |\nabla u|^{2n/(n+2)} \, dy \right)^{(n+2)/n} + \left(\fint_{\Omega_r(x)} |\nabla u|^t \, dy \right)^{2/t})].
$$

We let $s = \max(t, 2n/(n+2))$ and observe that our condition that $p > 2(n-1)/n$ implies that $t < 2$. We choose θ so that $1/s = (1-\theta)/1 + \theta/2$ and use Hölder's inequality and Young's inequality to obtain that

$$
\left(\fint_{\Omega_\rho(x)} |\nabla u|^2 \, dy \right)^{1/2} \leq C((1 - \rho/r)^{-\frac{2n-2}{1-\theta}} \fint_{\Omega_r(x)} |\nabla u| \, dy + P_r(|f|^p)^{1/p}(x))
$$
$$
+ \frac{1}{2} \left(\fint_{\Omega_r(x)} |\nabla u|^2 \, dy \right)^{1/2}
$$

Now a standard argument as in Giaquinta [15, p. 161] gives (3.6). ∎

4   Atomic Estimates

Our next step is to estimate the gradient of solutions at the boundary and eventually to obtain estimates for the non-tangential maximal function $(\nabla u)^*$.

It is this part of the argument that requires the integrability conditions on δ given in (2.2) and (2.3).

Our main result is Theorem 4.1. A key step in the proof, Lemma 4.10, uses estimates for the Green function from the work of Taylor, Kim, and Brown [32]. We are only able to prove these estimates in two dimensions.

Throughout this section, we assume that Ω is a Lipschitz domain in $\mathbf{R}^2$, that $D \subset \partial\Omega$ is an open set satisfying (2.1), and that L is the Lamé operator with coefficients satisfying (2.6). If q_0 is as in Theorem 3.1, we choose ϵ so that $1 < q_0(1-\epsilon)/(2-\epsilon)$ and require that Ω and D satisfy (2.2) and (2.3) for this ϵ.

Given a boundary ball $\Delta_r(x)$, we let $\Sigma_0 = S_r(x) = B_r(x) \cap \partial\Omega$ and for $k \geq 1$, we let $\Sigma_k = S_{2^k r}(x) \setminus S_{2^{k-1}r}(x)$. We will use this notation in the following proof and in a similar argument in Appendix B.

Theorem 4.1 *If u is a weak solution of the mixed problem (2.9) with $f = 0$ and traction data $f_N = a$ an atom supported in $\Delta_r(x)$, then for p with $p < q_0(1-\epsilon)/(2-\epsilon)$, we have*

$$\|(\nabla u)^*\|_{L^p(\Sigma_k)} \leq C 2^{-k\gamma} (2^k r)^{\frac{1}{p}-1}, \qquad k \geq 0. \tag{4.2}$$

Here, γ is the exponent of Hölder continuity for the Green function for our mixed problem (see [32] or the estimate (4.13) below). In addition, ∇u has non-tangential limits a.e. on $\partial\Omega$.

Let $p_1 = 1/(1+\gamma)$ then for $p_1 < p \leq 1$, we have

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C \sigma(\Delta_r(x))^{\frac{1}{p}-1}. \tag{4.3}$$

The constants in these estimates depend on p, M and the global properties of Ω.

We recall the following results of Dahlberg, Kenig, and Verchota [11] which are a consequence of the existence theory for the L^2-traction and regularity problems. These results do not depend on D and hold in all dimensions. To state these estimates, we will use the *truncated non-tangential maximal function* defined by

$$v_r^*(x) = \sup_{y \in \Gamma(x) \cap B_r(x)} |\nabla v(y)|, \qquad x \in \partial\Omega.$$

Lemma 4.4 ([11]) *Let $x \in \partial\Omega$ and $r \in (0, 25r_0)$. Suppose that u is a solution of $Lu = 0$ in $\Omega_{4r}(x)$ and u lies in the Sobolev space $W^{1,2}(\Omega_{4r}(x))$.*
If $\partial u/\partial\rho \in L^2(\Delta_{4r}(x))$, then

$$\int_{\Delta_r(x)} (\nabla u)_r^{*2} \, d\sigma \leq C \left(\int_{\Delta_{4r}(x)} \left|\frac{\partial u}{\partial\rho}\right|^2 \, d\sigma + \frac{1}{r} \int_{\Omega_{4r}(x)} |\nabla u|^2 \, dy \right).$$

If $\nabla_t u$ lies in $L^2(\Delta_{4r}(x))$, then

$$\int_{\Delta_r(x)} (\nabla u)_r^{*2} \, d\sigma \leq C \left(\int_{\Delta_{4r}(x)} |\nabla_t u|^2 \, d\sigma + \frac{1}{r} \int_{\Omega_{4r}(x)} |\nabla u|^2 \, dy \right).$$

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In each case, ∇u has non-tangential limits a.e. on $\Delta_r(x)$. The constants in these estimates depend only on the dimension and M.

These estimates may be proven using arguments from [11]. See Mayboroda and Mitrea [21] for results in two dimensions.

We give a notion of Whitney decomposition on the boundary of a Lipschitz domain. To pass from $\mathbf{R}^n$ to a more general set, we sacrifice the property that an open set is decomposed into disjoint sets and only require that each point lie in at most finitely many elements of the decomposition. This seems to be enough for our applications. Given an open set $\mathcal{O} \subset \partial\Omega$, a *Whitney decomposition* of $\mathcal{O}$ is a collection of boundary balls $\{\Delta_j = \Delta_{r_j}(x_j)\}_{j=1}^\infty$ so that 1) $\mathcal{O} = \cup_j \Delta_j$, 2) for a constant C_1 that may be chosen as large as we like, we have $C_1 r_j \leq \mathrm{dist}(\Delta_j, \partial\Omega \setminus \mathcal{O}) \leq 2C_1 r_j$, 3) $\chi_{\mathcal{O}} \leq \sum_j \chi_{\Delta_j} \leq c_n \chi_{\mathcal{O}}$, 4) $\sum_j \chi_{\Omega_{4r_j}(x_j)} \leq C$. We leave it as an exercise to construct this decomposition.

Lemma 4.5 *Suppose u is a weak solution of the mixed problem (2.9) with f_N in $L^2(N)$ and $f = 0$. Then for $x \in \partial\Omega$, $r \in (0, 50r_0)$ and $\rho \in \mathbf{R}$, we have*

$$\int_{\Delta_r(x)} (\nabla u)^*_{c\delta_r}{}^2 \delta_r^{1-\rho}\, d\sigma \leq C \left(\int_{\Delta_{2r}(x)} |f_N|^2 \delta_r^{1-\rho}\, d\sigma + \int_{\Omega_{2r}(x)} |\nabla u|^2 \delta_r^{-\rho}\, dy \right).$$

The constant in this estimate depends only on M, the dimension n, and ρ.

Proof. We construct a Whitney decomposition of $\partial\Omega \setminus \Lambda$ and let $\{\Delta_j = \Delta_{r_j}(x_j)\}$ denote the boundary balls from the Whitney decomposition which intersect $\Delta_r(x)$. This gives a family of boundary balls so that for each j, $\Delta_r(x) \setminus \Lambda \subset \cup_j \Delta_j \subset \Delta_{2r}(x)$, $\Delta_{2r_j}(x_j) \subset N$, or $\Delta_{2r_j}(x_j) \subset D$, $r_j \approx \delta_r(y)$ for $y \in \Omega_{4r_j}(x)$, and $\sum \chi_{\Omega_{4r_j}(x)} \leq C(M, n)$. We apply the estimate of Lemma 4.4, sum on j, and use the properties of the Whitney decomposition to obtain the Lemma. ∎

The next lemma takes us from estimates in a weighted L^2 space to estimates in an unweighted L^p space. Recall that ϵ appears in our hypotheses (2.2) and (2.3) and is required to satisfy $1 < q_0(1-\epsilon)/(2-\epsilon)$ with q_0 as in Theorem 3.1.

Lemma 4.6 *Let u be a weak solution of (2.9) with f_N in $L^\infty(N)$ and $f = 0$.*
Let $p_0 = q_0(1-\epsilon)/(2-\epsilon)$ and suppose that $1 < p < p_0$. For each boundary ball $\Delta_r(x)$ with $0 < r < 50r_0$.

$$\left(\fint_{\Delta_r(x)} (\nabla u)^*_{c\delta_r}{}^p\, d\sigma \right)^{1/p} \leq C \left(\fint_{\Omega_{2r}(x)} |\nabla u|\, dy + \|f_N\|_{L^\infty(\Delta_{2r}(x))} \right).$$

The constant in this local estimate depends on M and the constants in (2.2) and (2.3).

Proof. We choose p and q with $1 < p < q(1-\epsilon)/(2-\epsilon) < q_0(1-\epsilon)/(2-\epsilon)$. We then fix ρ so that $(2-\epsilon)(1-(2/q)) > \rho > (2-\epsilon) - 2(1-\epsilon)/p$. We apply Hölder's inequality with exponents $2/p$ and $2/(2-p)$, the property (2.2), the lower bound for ρ and Lemma 4.5 to obtain

$$\left(\fint_{\Delta_r(x)} (\nabla u)_{c\delta_r}^{*\;p}\, d\sigma \right)^{1/p} \le \left(\fint_{\Delta_r(x)} (\nabla u)_{c\delta_r}^{*\;2}\delta_r^{1-\rho}\, d\sigma \right)^{1/2} \left(\fint_{\Delta_r(x)} \delta_r^{(\rho-1)p/(2-p)}\, d\sigma \right)^{1/p-1/2}$$

$$\le Cr^{(\rho-1)/2} \left(\fint_{\Delta_r(x)} (\nabla u)_{c\delta_r}^{*\;2}\delta_r^{1-\rho}\, d\sigma \right)^{1/2}$$

$$\le C[r^{\rho/2} \left(\fint_{\Omega_{2r}(x)} |\nabla u|^2 \delta_r^{-\rho}\, dy \right)^{1/2}$$

$$+ r^{(\rho-1)/2} \left(\fint_{\Delta_{2r}(x)} |f_N|^2 \delta_r^{1-\rho}\, d\sigma \right)^{1/2}].$$

(4.7)

To estimate the integral over $\Omega_{2r}(x)$, we use Hölder's inequality with exponents $2/q$ and $q/(q-2)$, the property (2.3), the upper bound for ρ, and Theorem 3.1 to obtain

$$r^{\rho/2} \left(\fint_{\Omega_{2r}(x)} |\nabla u|^2 \delta_r^{-\rho}\, dy \right)^{1/2} \le C \left(\fint_{\Omega_{2r}(x)} |\nabla u|^q\, dy \right)^{1/q}$$

$$\le C[\fint_{\Omega_{4r}(x)} |\nabla u|\, dy + \left(\fint_{\Delta_{4r}(x)} |f_N|^{q(n-1)/n}\, d\sigma \right)^{\frac{n}{q(n-1)}}].$$

(4.8)

The Lemma follows from (4.7), (4.8), and a simple covering argument to replace $4r$ by $2r$ on the right hand side of the estimate. ∎

Before we proceed to the proof of Theorem 4.1, we recall a version of the energy estimate for solutions of the weak mixed problem, (2.9).

Lemma 4.9 *Let u be a solution of the weak mixed problem (2.9) with data $f_N = a$, an atom for N and $f = 0$. The solution u satisfies*

$$\int_\Omega |\nabla u|^2\, dy \le C.$$

The constant C depends on the global character of Ω.

Proof. According to (3.4), we have that $W_D^{1/2,2}(\partial\Omega) \subset BMO(\partial\Omega)$, the space of functions of bounded mean oscillation on $\partial\Omega$. Thus if a is an atom for N the map $u \to \int_{\partial\Omega} a^\beta u^\beta\, d\sigma$ lies in $W_D^{-1,2}(\Omega)$. The estimate of the Lemma follows. ∎

We are only able to prove the following Lemma in two dimensions. Finding an appropriate substitute for this Lemma is the main obstacle to studying the mixed problem for the Lamé system in higher dimensions.

Lemma 4.10 *Let a be an atom for N that is supported in a boundary ball $\Delta_r(x)$. We may find an exponent $\gamma > 0$ which depends only on M so that if u is a solution of the weak mixed problem (2.9) with $f_N = a$ and $f = 0$, then with p as in Theorem 4.1, we have a constant C so that*

$$\left(\int_{\Sigma_k} |\nabla u|^p \, d\sigma \right)^{1/p} \leq C 2^{-k\gamma} (2^k r)^{\frac{1}{p} - 1}, \qquad k \geq 0. \tag{4.11}$$

The constant in the estimate depends on the global character of Ω.

This Lemma gives the estimate of Theorem 4.1 for ∇u. Additional work is needed to obtain the estimate for the non-tangential maximal function.

Proof. We fix a boundary ball $\Delta_r(x)$ and let a be an atom for N that is supported in $\Delta_r(x)$. We let u be a solution of the weak mixed problem (2.9) with $f_N = a$ and $f = 0$. We apply Lemma 4.6 on a boundary ball $\Delta_R(y)$ and use the normalization of the atom to obtain that

$$\left(\fint_{\Delta_R(y)} |\nabla u|^p \, d\sigma \right)^{1/p} \leq C \left(\fint_{\Omega_{2R}(y)} |\nabla u| \, dz + \sigma(\Delta_r(x))^{-1} \right).$$

When $n = 2$, the Cauchy-Schwarz inequality and the energy estimate of Lemma 4.9 gives that for any local domain $\Omega_R(y)$

$$\fint_{\Omega_R(y)} |\nabla u| \, dz \leq \left(\fint_{\Omega_R(y)} |\nabla u|^2 \, dz \right)^{1/2} \leq C/R.$$

Together, these observations imply that

$$\left(\fint_{\Delta_R(y)} |\nabla u|^p \, d\sigma \right)^{1/p} \leq C(1/R + 1/r)$$

and if we set $y = x$, we obtain the estimate (4.11) but with a constant that depends on k.

We fix k_0 so that we have $\text{dist}(\Delta_r(x), \Sigma_{k_0}) \approx r$ and establish (4.11) for $k \geq k_0$ with constant that does not depend on k. Towards this end, we consider a boundary ball $\Delta_R(y)$ with $\Delta_{8R}(y) \cap \Delta_r(x) = \emptyset$. Since $a = 0$ on $\Delta_{2R}(y)$, the estimate of Lemma 4.6 implies that

$$\left(\fint_{\Delta_R(y)} |\nabla u|^p \, d\sigma \right)^{1/p} \leq C \fint_{\Omega_{2R}(y)} |\nabla u| \, dz.$$

The local ellipticity condition (2.8) implies a Caccioppoli inequality

$$\int_{\Omega_{2R}(y)} |\nabla u|^2 \, dz \leq \frac{C}{R^2} \int_{\Omega_{4R}(y)} |u|^2 \, dz \tag{4.12}$$

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provided that u is a solution of (2.9) with $f_N = 0$ on $\Delta_{4R}(y)$ and $f = 0$ on $\Omega_{4R}(y)$. Next we note that the Green function estimates of Taylor, Kim, and Brown [32] and the arguments in [26] imply that we have a constant C and exponent $\gamma > 0$ so that

$$|u(y)| \le C \left(\frac{r}{|x-y|} \right)^{\gamma}, \qquad y \in \Omega \setminus \Omega_{2r}(x). \tag{4.13}$$

Combining the estimate of Lemma 4.6, the Hölder inequality, (4.12), and (4.13) gives that

$$\left(\fint_{\Delta_R(y)} |\nabla u|^p \, d\sigma \right)^{1/p} \le \frac{C}{R} \left(\frac{r}{R} \right)^{\gamma}.$$

We may cover Σ_k by boundary balls with radius comparable to $2^k r$ and use the above estimate to obtain the estimate of the Lemma for $k \ge k_0$. ∎

Now we are ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1. We let $\phi \in C^\infty(\bar{\Omega})$ be a smooth function which is zero outside $\Omega_{4r}(x)$ and is zero in a neighborhood of Λ. We let $\Gamma^{\alpha\beta}$ be the matrix fundamental solution of the Lamé operator in $\mathbf{R}^2$, thus $(L\Gamma^{\cdot\gamma})^\alpha = \delta_{\alpha\gamma}\delta_0$ where $\delta_{\alpha\gamma}$ is the Kronecker delta and δ_0 is the Dirac delta measure at 0. We claim the following representation formula for derivatives of u, with $\alpha, k = 1, 2$,

$$
\begin{aligned}
\phi(x)\frac{\partial u^\alpha}{\partial x_k}(x) &= \int_\Omega \frac{\partial}{\partial y_i} a^{ij}_{\alpha\beta} \frac{\partial \Gamma^{\beta\gamma}}{\partial y_j}(x - \cdot)\phi \frac{\partial u^\alpha}{\partial y_k} \, dy \\
&= \int_{\partial\Omega} \nu_i a^{ij}_{\alpha\beta} \frac{\partial \Gamma^{\beta\gamma}}{\partial y_j}(x - \cdot)\phi \frac{\partial u^\alpha}{\partial y_k} - \nu_k a^{ij}_{\alpha\beta} \frac{\partial \Gamma^{\beta\gamma}}{\partial y_j}(x - \cdot)\phi \frac{\partial u^\alpha}{\partial y_i} \\
&\quad + \nu_j a^{ij}_{\alpha\beta} \frac{\partial \Gamma^{\beta\gamma}}{\partial y_k}(x - \cdot)\phi \frac{\partial u^\alpha}{\partial y_i} \, d\sigma - \int_\Omega a^{ij}_{\alpha\beta} \frac{\partial \Gamma^{\beta\gamma}}{\partial y_j}(x - \cdot)\frac{\partial \phi}{\partial y_i}\frac{\partial u^\alpha}{\partial y_k} \\
&\quad - a^{ij}_{\alpha\beta} \frac{\partial \Gamma^{\beta\gamma}}{\partial y_j}(x - \cdot)\frac{\partial \phi}{\partial y_k}\frac{\partial u^\alpha}{\partial y_i} + a^{ij}_{\alpha\beta} \frac{\partial \Gamma^{\beta\gamma}}{\partial y_k}(x - \cdot)\frac{\partial \phi}{\partial y_j}\frac{\partial u^\alpha}{\partial y_i} \, d\sigma.
\end{aligned}
\tag{4.14}
$$

Of course, the integral to the right of the first equals sign should be interpreted as the Dirac delta acting on a smooth function. If u is smooth, the proof of (4.14) is accomplished by integration by parts. We begin by moving the $\partial/\partial y_i$ derivative, then the $\partial/\partial y_k$ derivative, and finally the $\partial/\partial y_j$ derivative. Since Lemma 4.4 gives non-tangential maximal function estimates for u and ∇u away from Λ and we assume that ϕ is supported in a coordinate cylinder and is zero in a neighborhood of Λ, standard limiting arguments allow us to obtain the representation formula (4.14) without the assumption that u is smooth in $\bar{\Omega}$.

Next, we remove the restriction that ϕ vanish near Λ. Let ψ_τ be a function which is one when $\delta(x) > 2\tau$, zero if $\delta(x) < \tau$, and satisfies $|\partial^\alpha \psi_\tau/\partial x^\alpha| \le C_\alpha/\tau^{|\alpha|}$. Replace ϕ in (4.14) by $\psi_\tau \phi$ where ϕ is a smooth function which is zero

outside $\Omega_{4r}(x)$, but is not required to vanish on Λ. Using Hölder's inequality, the terms involving derivatives of ψ_τ in (4.14) have the upper bound

$$\int_{\Lambda_{\tau,r}} |\nabla\Gamma(x-\cdot)||\nabla u||\phi\nabla\psi_\tau|\,dy \leq \frac{C_x}{\tau}\left(\int_{\Lambda_{\tau,r}} |\nabla u|^q\,dy\right)^{1/q}|\Lambda_{\tau,r}|^{1/q'} \quad (4.15)$$

where $\Lambda_{\tau,r} = \Omega_{4r}(x)\cap\{y : \tau < \delta(y) < 2\tau\}$ and q is chosen to satisfy $(2-\epsilon)/(1-\epsilon) < q < q_0$. From the estimate (2.3), we obtain the upper bound

$$|\Lambda_{\tau,r}| \leq C\tau^a\int_{\Omega_{4r}(x)}\delta^{-a}\,dy \leq C_r\tau^a$$

provided $0 < a < 2-\epsilon$. Our choice of q implies $q' < 2-\epsilon$ so we set $a = q'$. Then the estimate (3.2) for ∇u implies that the right-hand side of (4.15) tends to zero with τ. From the estimate of Lemma 4.10, we have ∇u is in $L^p(\partial\Omega)$ for some $p \geq 1$, thus we may use the dominated convergence theorem and let τ tend to zero in the boundary terms of (4.14). This removes the restriction that ϕ vanish in a neighborhood of Λ.

To establish (4.2), we begin with the representation formula (4.14). We let ϕ run over a partition of unity and sum to obtain the representation

$$\frac{\partial u^\alpha}{\partial x_k}(x) = \int_{\partial\Omega} a_{\alpha\beta}^{ij}\frac{\partial\Gamma^{\beta\gamma}}{\partial y_j}(x-\cdot)(\nu_i\frac{\partial u^\alpha}{\partial y_k} - \nu_k\frac{\partial u^\alpha}{\partial y_i}) + \frac{\partial\Gamma^{\beta\gamma}}{\partial y_k}(x-\cdot)\nu_j a_{\alpha\beta}^{ij}\frac{\partial u^\alpha}{\partial y_i}\,d\sigma, \alpha, k = 1, 2.$$

The divergence theorem implies that

$$\int_{\partial\Omega}\nu_i a_{\alpha\beta}^{ij}\frac{\partial u^\beta}{\partial y_j}\,d\sigma = 0, \qquad \alpha = 1, 2 \qquad\qquad (4.16)$$

$$\int_{\partial\Omega}\nu_i\frac{\partial u^\beta}{\partial y_k} - \nu_k\frac{\partial u^\beta}{\partial y_i}\,d\sigma = 0, \qquad i, k, \beta = 1, 2 \qquad\qquad (4.17)$$

where the techniques used to prove (4.14) help to justify the application of the divergence theorem. We may use the theorem of Coifman, McIntosh, and Meyer [6], the observations (4.16-4.17), the size estimates of Lemma 4.10, and the notion of molecule (see [7]) to establish the estimates (4.2). For $p > 1/(1+\gamma)$, the estimate (4.3) follows easily from (4.2). ∎

5 Uniqueness

We take advantage of the restriction to $n = 2$ to give a different proof of uniqueness than found in [26, 33]. The argument uses a version of the Hardy-Littlewood-Sobolev theorem on fractional integration. The proof is adapted from one found in [3] and [24, Lemma 11.9] where a result similar to Lemma 5.1 is established in dimensions $n \geq 3$. The result for $n = 2$ is more technically involved.

Lemma 5.1 *Let L be the Lamé operator with coefficients satisfying (2.6). Let $\Omega \subset \mathbf{R}^2$ be a Lipschitz domain and suppose that u is a solution of $Lu = 0$ with $(\nabla u)^* \in L^p(\partial\Omega)$. If $p < 1$, q is defined by $1/q = 1/p - 1$, and K is a compact subset of Ω, then we have*

$$\|u^*\|_{L^q(\partial\Omega)} \leq C(\|(\nabla u)^*\|_{L^p(\partial\Omega)} + r_0^{1/p-1} \sup_K |u|).$$

The constant in this estimate depends on p, K and the global character of Ω.

For this argument it will be convenient to use $S_r(x) = B_r(x) \cap \partial\Omega$ for $x \in \partial\Omega$ in place of $\Delta_r(x)$ and to define the Hardy-Littlewood maximal operator on the boundary by

$$\mathcal{M}f(x) = \sup_{r>0} \fint_{S_r(x)} |f| \, d\sigma.$$

Proof. We let $\{\Omega_k\}$ be a sequence of approximating domains with $\bar{\Omega}_k \subset \Omega$ as in Verchota [36]. We will show that we have

$$\|u^*\|_{L^q(\partial\Omega_k)} \leq C(\|(\nabla u)^*\|_{L^p(\partial\Omega_k)} + r_0^{1/q} \sup_K |u|) \tag{5.2}$$

where p and q are as in the statement of the Lemma and the constant C is independent of k. Letting k tend to ∞ gives the result for Ω. The advantage of working on the subdomain Ω_k is that we may assume that $\|u^*\|_{L^q(\partial\Omega_k)}$ is finite. In what follows, we drop the subscript k and assume that $\|u^*\|_{L^q(\partial\Omega)}$ is finite.

To begin we observe that if $y \in \Gamma(x)$, for some $x \in \partial\Omega$, then for $\epsilon > 0$, and β a multi-index, we have

$$\left| \frac{\partial^\beta u}{\partial y^\beta}(y) \right| \leq \frac{C}{d(y)^{|\beta|}} \left(\fint_{S(x,y)} u^{*\epsilon} \, d\sigma \right)^{1/\epsilon}. \tag{5.3}$$

Here, $S(x,y) = \{z : |z - x| < (2 + 2\alpha)d(y)\} \cap \partial\Omega$ with α as in the definition of the approach regions $\Gamma(x)$ and recall that $d(y) = \text{dist}(y, \partial\Omega)$. To establish (5.3), choose $\hat{x} \in \partial\Omega$ so that $|\hat{x} - y| = d(y)$. We claim that if $|w - y| < \min\{\alpha/4, \alpha/(2 + 2\alpha)\}d(y)$ and $|z - \hat{x}| < \alpha d(y)/4$, then $w \in \Gamma(z)$. To establish this claim, first note that $d(y) \leq d(w) + |w - y| \leq d(w) + \alpha d(y)/(2 + 2\alpha)$ and hence $d(y) \leq d(w)(2 + 2\alpha)/(2 + \alpha)$. Using the triangle inequality and our choices for z, $\hat{x}$, and w, we have $|z - w| \leq |w - y| + |y - \hat{x}| + |\hat{x} - z| < (1 + \alpha/2)d(y) \leq (1 + \alpha)d(w)$. Also, if $z \in S_{\alpha d(y)/4}(\hat{x})$, then by the triangle inequality $|z - x| \leq |z - \hat{x}| + |\hat{x} - y| + |y - x| < (2 + 2\alpha)d(y)$ and we conclude that $S_{\alpha d(y)/4}(\hat{x}) \subset S_{(2+2\alpha)d(y)}(x)$. Thus, with $r = \min\{\alpha/4, \alpha/(2 + 2\alpha)\}d(y)$, we may use interior estimates for derivatives of u to obtain

$$\left| \frac{\partial^\beta u}{\partial y^\beta}(y) \right| \leq \frac{C}{d(y)^{|\beta|}} \sup_{w \in B_r(y)} |u(w)| \leq \frac{C}{d(y)^{|\beta|}} \left(\fint_{S(x,y)} u^{*\epsilon} \, d\sigma \right)^{1/\epsilon}.$$

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The second inequality follows since $B_r(y) \subset \Gamma(z)$ for all z in $S_{\alpha d(y)/4}(x)$ and hence $S_{\alpha d(y)}(x) \subset S(x, y)$. We have established (5.3).

To begin the proof of (5.2), we let $\{Z_i = Z_{r_0}(x_i)\}_{i=1}^N$ be the covering of $\partial\Omega$ by coordinate cylinders guaranteed by the definition of a Lipschitz domain and recall that we are in $\mathbf{R}^2$. We let $T_i = \{(y_1, y_2) : |y_1 - x_{i,1}| \le r_0, y_2 = x_{i,2} + (4M + 2)r_0\}$ denote the "top" of each cylinder Z_i and then let K' denote the compact set $\cup_i T_i$. Applying the estimate (5.3) to ∇u with $\beta = 0$ and $\epsilon = p$ gives that for any pair K and J of compact subsets of Ω, we have

$$\sup_J |u| \le \sup_K |u| + Cr_0^{1-1/p}\|(\nabla u)^*\|_{L^p(\partial\Omega)}. \tag{5.4}$$

The heart of the proof is to establish for $\epsilon > 0$, $\theta \in (0, 1)$, and $x \in \partial\Omega$, that

$$|u^*(x)| \le C(\mathcal{M}(u^{*\epsilon})(x)^{\frac{1-\theta}{\epsilon}} I_\theta((\nabla u)^*)(x)^\theta + \sup_{K'} |u| + r_0^{1-\frac{1}{p}}\|(\nabla u)^*\|_{L^p(\partial\Omega)}) \tag{5.5}$$

where I_θ is the fractional integral defined by

$$I_\theta f(x) = \int_{\partial\Omega} f(y)|x - y|^{\theta-1}\, d\sigma.$$

We accept the claim (5.5) for the moment and complete the proof of the Theorem. We raise both sides of (5.5) to the power q, integrate over $\partial\Omega$, and apply the Hölder inequality with exponents $1/\theta$ and $1/(1 - \theta)$ to obtain

$$\|u^*\|_{L^q(\partial\Omega)} \le C(\|\mathcal{M}(u^{*\epsilon})\|_{L^{q/\epsilon}(\partial\Omega)}^{(1-\theta)/\epsilon}\|I_\theta((\nabla u)^*)\|_{L^q(\partial\Omega)}^\theta + r_0^{1/q}\sup_{K'}|u| + \|(\nabla u)^*\|_{L^p(\partial\Omega)})$$

We choose ϵ so that $q/\epsilon > 1$ and hence the maximal operator $\mathcal{M}$ is bounded on $L^{q/\epsilon}(\partial\Omega)$, choose θ with $0 < \theta < p$ so that $I_\theta : L^{p/\theta}(\partial\Omega) \to L^{q/\theta}(\partial\Omega)$ with $\theta/q = \theta(1/p - 1)$. With these choices, we may use Young's inequality, our *a priori* assumption that $u^* \in L^q(\partial\Omega)$ and (5.4) to obtain the Theorem.

Finally, we establish the key estimate (5.5). We fix $x \in \partial\Omega$ and consider $y \in \Gamma(x)$ and suppose that y lies in a coordinate cylinder Z_i. If α is large enough, we have that the line segment $\{y + te_2 : y_2 < t < r_0\}$ lies in $\Gamma(x)$. (As usual, we are using the coordinate system for the coordinate cylinder Z_i to state this condition.) Thus, we may use the fundamental theorem of calculus to write

$$|u(y)| \le \int_{y_2}^{r_0} |\frac{\partial u}{\partial y_2}(y_1, t)|\, dt + \sup_{K'} |u|. \tag{5.6}$$

We apply the estimate (5.3) to obtain the bound

$$|\frac{\partial u}{\partial y_2}(y_1, t)|^{1-\theta} \le \frac{C}{d(y_1, t)^{1-\theta}}\mathcal{M}(u^{*\epsilon})(x)^{(1-\theta)/\epsilon}.$$

Also, if we apply estimate (5.3) to ∇u, then we obtain

$$|\nabla u(y_1, t)|^\theta \le C \fint_{S(x,(y_1,t))} (\nabla u)^{*\theta}\, d\sigma.$$

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We write $|\partial u/\partial y_2|^{1-\theta+\theta}$ in (5.6) and use the two previous estimates to obtain

$$\int_{y_2}^{r_0} |\frac{\partial u}{\partial y_2}(y_1, t)| \, dt \leq C\mathcal{M}(u^{*\epsilon})(x)^{(1-\theta)/\epsilon}$$

$$\times \int_{y_2}^{r_0} \frac{1}{d((y_1, t))^{2-\theta}} \int_{S(x,(y_1,t))} (\nabla u)^{*\theta} \, d\sigma \, dt \qquad (5.7)$$

$$\leq C\mathcal{M}(u^{*\epsilon})(x)^{(1-\theta)/\epsilon} I_\theta((\nabla u)^{*\theta})(x).$$

The last step uses Tonelli's theorem to change the order of integration. The estimates (5.6) and (5.7) give a bound for the supremum of u in $\Gamma(x) \cap (\cup Z_i)$. Estimate (5.4) allows us to estimate the supremum of u in $\Omega \setminus \cup_i Z_i$. Together these two observations give (5.5). This completes the proof of the estimate (5.5) and hence the Lemma. ∎

We are ready to give uniqueness for the L^p-mixed problem.

Theorem 5.8 *Suppose $\Omega \subset \mathbf{R}^2$ is Lipschitz domain, Ω and D satisfy (2.1), (2.2), and (2.3) with $q_0(1-\epsilon)/(2-\epsilon) > 1$, and L is the Lamé system with coefficients satisfying (2.6). There exist $p_1 < 1$ so that if $p > p_1$ and u is a solution of (1.1) with $f_N = 0$ and $f_D = 0$, then $u = 0$. We assume that the traction operator exists in the sense of non-tangential limits if $p \geq 1$ and in the sense of (2.11) if $p < 1$.*

Proof. It suffices to prove the Theorem for $p < 1$. We let u be a solution of the mixed problem (1.1) with $f_N = 0$ and $f_D = 0$. Our argument will proceed by duality and requires the existence of a solution when the traction data is an atom for N as given in Theorem 4.1. We let a be an atom for N and claim that

$$\int_N a^\beta u^\beta \, d\sigma = 0. \qquad (5.9)$$

We accept the claim and complete the proof. We first observe that $(\nabla u)^* \in L^p(\partial\Omega)$ and thus Lemma 5.1 implies that $u^* \in L^q(\partial\Omega)$ with $1/q = 1/p - 1$. Also, it is easy to see that if $(\nabla u)^* \in L^p(\partial\Omega)$ then u has non-tangential limits a.e. The claim implies that these limits are zero. From the work of [11], we have uniqueness for the L^q-Dirichlet problem for $q \geq 2$ and any Lipschitz domain and thus we have $u = 0$ if $q \geq 2$. Since we are in two dimensions, we have $q \geq 2$ if $p \geq 2/3$, thus we will require that $p_1 \geq 2/3$.

To establish the claim (5.9), we let v be the solution of (1.1) with data $f_N = a$ and $f_D = 0$. From Theorem 4.1 we have $(\nabla v)^* \in L^t(\partial\Omega)$ for $t < p_0$. Lemma 5.1 implies that $v^* \in L^q(\partial\Omega)$ for $1/q = 1/t - 1$. Thus if $1/p_0 + 1/p_1 \leq 2$, then we may find t so that $1/t + 1/p \leq 1$ and use the dominated convergence theorem to obtain

$$\lim_{k \to \infty} \int_{\partial\Omega_k} u^\beta (\frac{\partial v}{\partial \rho})^\beta \, d\sigma = \int_N u^\beta a^\beta \, d\sigma.$$

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The estimates for the Green function in section 4 of [32] imply that there is an exponent γ so that the solution v lies in $C_D^\alpha(\partial\Omega)$ for $\alpha \le \gamma$. If $\alpha \ge 1/p_1 - 1$, then our assumption that $\partial u/\partial\rho = 0$ in $H^p(N)$ implies that

$$\lim_{k\to\infty} \int_{\partial\Omega_k} v^\beta \left(\frac{\partial u}{\partial\rho}\right)^\beta d\sigma = 0.$$

While the Green identity gives for every k that

$$\int_{\partial\Omega_k} v^\beta \left(\frac{\partial u}{\partial\rho}\right)^\beta - u^\beta \left(\frac{\partial v}{\partial\rho}\right)^\beta d\sigma = 0.$$

The last three displayed equations imply the claim (5.9).

To summarize the conditions on p_1, we need $p_1 \ge 2/3$, $p_1 \ge p_0/(2p_0 - 1)$ and $p_1 \ge 1/(1 + \gamma)$ to establish uniqueness. ∎

6 Existence of solutions

In this section, we give the details needed to establish the existence of solutions for the L^p-mixed problem for p in an interval containing 1. When $p \le 1$, we take our data from Hardy spaces and for $p > 1$, the data is taken from L^p spaces. Given the estimates of Theorem 4.1, the argument is not so different from results in previous work of the authors [3] and [26].

Theorem 6.1 *Let $\Omega \subset \mathbf{R}^2$ be a Lipschitz domain, suppose that D satisfies (2.1), (2.2), and (2.3), and that L is the Lamé operator with coefficients satisfying the ellipticity condition (2.6).*

We may find an exponent p_1 so that if $1 \ge p > p_1$, the L^p-mixed problem has a solution. This means that if $f_N \in H^p(N)$ and $f_D \in H^{1,p}(\partial\Omega)$, then we have a solution to the L^p-mixed problem (1.1) which satisfies

$$\|u\|_{H^{1,p}(\partial\Omega)} + \|\frac{\partial u}{\partial\rho}\|_{H^p(\partial\Omega)} + \|(\nabla u)^*\|_{L^p(\partial\Omega)} \le C(\|f_N\|_{H^p(N)} + \|f_D\|_{H^{1,p}(\partial\Omega)}).$$

The normal derivative of u exists in the sense of (2.11), u has non-tangential limits a.e. on the boundary, and these limits vanish on D.

Proof.   By the results of Appendix B, we may assume that $f_D = 0$. Suppose that f_N lies in $H^p(N)$ and choose a representation of $f_N = \sum_j \lambda_j a_j$ with the atom a_j supported in $\Delta_{r_j}(x_j)$ and $\sum_j \lambda_j^p \sigma(\Delta_{r_j}(x_j))^{1-p} \le 2\|f_N\|_{H^p(N)}$. We let v_j be the solution of (2.9) with the traction data f_N an atom a_j and $f = 0$ and set $u_M = \sum_{j=1}^M \lambda_j v_j$. From Theorem 4.1, we have

$$\|\nabla(u_M - u_{M'})^*\|_{L^p(\partial\Omega)}^p \le C \sum_{j=M+1}^{M'} \lambda_j^p \sigma(\Delta_{r_j}(x_j))^{1-p}, \qquad M' > M.$$

From Lemma 5.1, we have that $u_M^* \in L^q(\partial\Omega)$ for $1/q = 1/p - 1$. Thus, the estimate of (5.3) implies that u_M converges uniformly to a function u on compact subsets of Ω. Furthermore, we have that u is a solution of $Lu = 0$ in Ω, $\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|f_N\|_{H^p(N)}$, and ∇u has non-tangential limits a.e. on $\partial\Omega$. To find $\partial u/\partial\rho$ on $\partial\Omega$, we first note that as in (4.16), we have that for any j and k,

$$\int_{\partial\Omega} \frac{\partial v_j}{\partial\rho}\, d\sigma = 0 \qquad \text{and} \qquad \int_{\partial\Omega_k} \frac{\partial v_j}{\partial\rho}\, d\sigma = 0.$$

The estimates (4.2) of Theorem 4.1 imply that

$$\|\frac{\partial v_j}{\partial\rho}\|_{H^p(\partial\Omega)} + \sup_k \|\frac{\partial v_j}{\partial\rho}\|_{H^p(\partial\Omega_k)} \leq C. \tag{6.2}$$

To establish (6.2), we observe that the size estimates (4.2) and (4.16) imply that $\partial v_j/\partial\rho$ is a molecule as in [7] and thus can be decomposed into atoms. Here, we use that the atomic Hardy spaces may be defined with atoms from any L^t space, $t > 1$, and the space we obtain is independent of t. See Coifman and Weiss [7] or Mitrea and Wright [24, sections 2.2-2.3] for more information. Thus $\sum_{j=1}^{\infty} \lambda_j \partial v_j/\partial\rho$ defines an element of the atomic Hardy space $H^p(\partial\Omega)$ as long as $1/(1+\gamma) < p \leq 1$ with γ as in (4.2). It is also straightforward to see that the estimate (4.2) implies that $\sum_{j=1}^{\infty} \lambda_j v_j$ defines an element of the Hardy-Sobolev space $H^{1,p}(\partial\Omega)$. Now we assume $p < 1$, fix $\phi \in C^\alpha(\bar\Omega)$ with $\alpha = \frac{1}{p} - 1$, and consider the limit

$$\left| \lim_{k\to\infty} \int_{\partial\Omega_k} \frac{\partial u}{\partial\rho}\phi\, d\sigma - \sum_{j=1}^{\infty} \lambda_j \int_{\partial\Omega} \frac{\partial v_j}{\partial\rho}\phi\, d\sigma \right|$$

$$\leq \limsup_{k\to\infty} \sum_{j=1}^{M} \lambda_j \left| \int_{\partial\Omega_k} \frac{\partial v_j}{\partial\rho}\phi\, d\sigma - \int_{\partial\Omega} \frac{\partial v_j}{\partial\rho}\phi\, d\sigma \right|$$

$$+ \sup_k \sum_{j=M+1}^{\infty} \lambda_j \left(| \int_{\partial\Omega_k} \frac{\partial v_j}{\partial\rho}\phi\, d\sigma| + | \int_{\partial\Omega} \frac{\partial v_j}{\partial\rho}\phi\, d\sigma| \right).$$

Using the estimate (4.2) for the non-tangential maximal function of ∇v_j and the dominated convergence theorem, we see that the first term on the right of this inequality is zero for any M. Since functions in C^α, $\alpha = 1/p - 1$ give rise to elements of the dual of H^p, the second and third terms are small when M is large by our choice of λ_j and the estimate for $\partial v_j/\partial\rho$ in (6.2). Hence we have that $\partial u/\partial\rho$ exists in the sense of (2.11) and is given by $\sum \lambda_j \partial v_j/\partial\rho$ and we have the estimate for $\partial u/\partial\rho$ in the estimate of the Theorem. Note that we also obtain the estimate when $p = 1$. Finally, if we restrict ϕ to lie in $C_D^\alpha(\bar\Omega)$, we have that $\partial v_j/\partial\rho = a_j$ on the set N and thus we have $\partial u/\partial\rho = f_N$. ∎

Finally, we observe that a real-variable argument of Caffarelli and Peral [5] and Shen [28] gives existence for $1 < p < p_0$. The argument to obtain this result is identical to that used in our study of the Laplacian in our earlier work with Taylor [26, 33].

Theorem 6.3 *Let Ω be a Lipschitz domain and let $D \subset \partial\Omega$ be a non-empty proper open subset of $\partial\Omega$ which satisfies (2.1), (2.2), and (2.3). Suppose that $1 < p < p_0$ with p_0 as in Theorem 4.1. We assume that the coefficients of the operator L satisfy (2.6). Let $f_N \in L^p(N)$ and $f_D = 0$, then we may find a solution of the L^p-mixed problem (1.1) which satisfies*

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|f_N\|_{L^p(N)}.$$

The following lemma is a key estimate that is needed to carry out the method of Shen [28]. This result may be proved using the techniques in the proof of Theorem 4.1. See also the argument in section 6 of our previous work [26].

Lemma 6.4 *Let Ω, D, and L be as in Theorem 6.3. Suppose that $1 < p < p_0 = q_0(1 - \epsilon)/(2 - \epsilon)$.*

If u is a weak solution of (2.9) with data f_N in $L^p(N)$ and $f = 0$, we have the following local estimate for $1 < p < p_0$,

$$\left(\fint_{\Delta_r(x)} (\nabla u)_r^{*p}\, d\sigma\right)^{1/p} \leq C\left(\fint_{\Omega_{4r}(x)} |\nabla u|\, dy + \left(\fint_{\Delta_{2r}(x)\cap N} |f_N|^p\, d\sigma\right)^{1/p}\right). \quad (6.5)$$

The constant C depends only on M and the exponent p.

A   Sobolev inequalities

In this appendix, we establish the estimates (3.3) and (3.4) that were used in the study of the mixed problem. These results may be found in our earlier work [26, 32, 33], though we do not claim originality. The exposition below serves to collect these results in one place and includes an occasional endpoint that was missed in our earlier work.

Let ϕ be a Lipschitz function on the unit sphere so that $\Omega = \{y : |y| < r\phi(y/|y|)\}$ is a star-shaped Lipschitz domain of scale r and centered at 0. We define a bi-Lipschitz map $\Phi : B_1(0) \to \Omega$ by

$$\Phi(y) = r\phi(y/|y|)y. \quad (A.1)$$

Using the argument in [17, Lemma 7.16] and a change of variables, we may find a constant $C = C(N, n, p)$ so that for $S \subset \Omega$, a set of positive Lebesgue measure and $1 \leq p < \infty$, we have

$$\left\|u - \fint_S u\, dy\right\|_{L^p(\Omega)} \leq \frac{C(M, n, p)r^{n+1}}{|S|}\|\nabla u\|_{L^p(\Omega)}. \quad (A.2)$$

The only change that is needed from the standard argument is to average u with respect to the weight given by the Jacobian of the change of variables.

Next we observe that for Ω a star-shaped convex domain with constant M and scale r, $1 \leq p < n$, q defined by $1/q = 1/p - 1/n$, and S a measurable subset of Ω, we have the Sobolev-Poincaré inequality

$$\|u - \fint_S u \, dy\|_{L^q(\Omega)} \leq \frac{C(M,p,n)r^n}{|S|} \|\nabla u\|_{L^p(\Omega)}. \qquad (A.3)$$

To establish (A.3) we extend u to $\mathbf{R}^n$ by a reflection in $\partial\Omega$. Thus choose η a cutoff function which is one when $|x| \leq (1+M)r$ and zero when $|x| > (2+2M)r$. We let

$$Eu(x) = \begin{cases} u(x), & x \in \Omega \\ u(xr^2\phi(x/|x|)^2/|x|^2)\eta(x), & x \in \mathbf{R}^n \setminus \Omega. \end{cases}$$

The Sobolev inequality and the product rule in $\mathbf{R}^n$ gives

$$\begin{aligned}
\|u - \fint_S u \, dy\|_{L^q(\Omega)} &\leq \|E(u - \fint_S u \, dy)\|_{L^q(\mathbf{R}^n)} \\
&\leq C(p,n)\|\nabla E(u - \fint_S u \, dy)\|_{L^p(\mathbf{R}^n)} \\
&\leq C(\|\nabla u\|_{L^p(\Omega)} + r^{-1}\|u - \fint_S u \, dy\|_{L^p(\Omega)}).
\end{aligned}$$

Applying the estimate (A.2) gives the Sobolev-Poincaré inequality (A.3).

The following estimate was proved in [26] under an additional assumption that the set D was not too spread out. The proof below is simpler and omits that assumption.

Lemma A.4 *If Ω is a star-shaped Lipschitz domain with scale r and constant M, D a measurable subset of $\partial\Omega$, and $u \in W_D^{1,p}(\Omega)$, then for $1 \leq p < n$ and q given by $1/q = 1/p - 1/n$, we have*

$$\left(\fint_\Omega |u|^q \, dy\right)^{1/q} \leq C\frac{r^{n-1}}{\sigma(D)}\left(\fint_\Omega |\nabla u|^p \, dy\right)^{1/p}$$

The constant C depends on n, p, and M.

Proof. It suffices to prove this estimate for a function u in $C^\infty(\bar{\Omega})$ which vanishes in a neighborhood of D. Using the map Φ defined in (A.1), we may reduce to considering the domain $\Omega = B_1(0)$. We set $A = \{y \in \partial B_1(0) : \Phi(y) \in D\}$, then we have

$$C(M)\sigma(D) \leq \sigma(A)r^{n-1} \leq \sigma(D).$$

We will show that if $u \in C^\infty(\bar{B}_1(0))$ and u vanishes on $A \subset \partial B_1(0)$, then for $1 \leq p < n$

$$\left(\fint_{B_1(0)} |u|^{np/(n-p)} \, dy\right)^{1/p-1/n} \leq \frac{C}{\sigma(A)}\left(\fint_{B_1(0)} |\nabla u|^p \, dy\right)^{1/p}. \qquad (A.5)$$

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As the Lemma follows from the special case where Ω is the unit ball, $B_1(0)$, we only need to prove the claim (A.5).

We let $\tilde{A} = \{y \in B_1(0) : y/|y| \in A\}$ and observe that

$$\left| \int_{\tilde{A}} u(y)\, dy \right| \leq \frac{1}{n} \int_{\tilde{A}} |y \cdot \nabla u(y)|\, dy. \qquad (A.6)$$

To establish (A.6), we let s be in $[0, 1]$, $y \in A$ and use the fundamental theorem of calculus to write

$$u(sy) = -\int_s^1 \frac{d}{dt} u(ty)\, dt = -\int_s^1 y \cdot \nabla u(ty)\, dt.$$

We multiply by s^{n-1} and integrate on $[0, 1] \times A$ to obtain

$$\int_{\tilde{A}} u(y)\, dy = \int_A \int_0^1 u(sy) s^{n-1}\, ds\, d\sigma = -\frac{1}{n} \int_{\tilde{A}} z \cdot \nabla u(z)\, dz$$

which implies (A.6). Given (A.6), the claim (A.5) follows from (A.3). ∎

We are now ready to establish (3.3). Observe that if $\mathrm{dist}(\Omega_\rho(x), D) > 0$, then $\bar{u}_{x,\rho} = \fint_{\Omega_\rho(x)} u\, dy$ and thus (3.3) follows from (A.3) and in this case, the constant remains bounded as ρ approaches r. If $\bar{u}_{x,\rho} = 0$, then we have $\mathrm{dist}(\Omega_\rho(x), D) = 0$. In this case, the corkscrew condition (2.1) implies that $\sigma(D \cap \bar{\Omega}_r(x)) \geq c(r - \rho)^{n-1}$ and (3.3) follows from Lemma A.4.

To establish the inequality (3.4) we work in a coordinate cylinder and let $\eta(x', x_n)$ be a cut off function which is 1 for $|x_n - \phi(x')| < \rho/2$ and 0 for $|x_n - \phi(x')| > \rho$. We apply the fundamental theorem of calculus and obtain that

$$-\int_{\Delta_\rho(x)} |u - \bar{u}_{x,\rho}|^q e_n \cdot \nu\, d\sigma = \int_{\Omega_\rho(x)} q \frac{(u - \bar{u}_{x,\rho})^\alpha}{|u - \bar{u}_{x,\rho}|} \frac{\partial u^\alpha}{\partial y_n} |u - \bar{u}_{x,\rho}|^{q-1} \eta + |u - \bar{u}_{x,\rho}|^q \frac{\partial \eta}{\partial y_n}\, dy. \qquad (A.7)$$

We apply the Hölder inequality and obtain that the following is an upper bound to the right-hand side of (A.7),

$$q \left(\int_{\Omega_\rho(x)} \left| \frac{\partial u}{\partial y_n} \right|^p dy \right)^{1/p} \left(\int_{\Omega_\rho(x)} |u - \bar{u}_{x,\rho}|^{(q-1)p'} dy \right)^{1/p'} + C \left(\int_{\Omega_\rho(x)} |u - \bar{u}_{x,\rho}|^{qn/(n-1)} dy \right)^{(n-1)/n}.$$

If we have the relation $(q - 1)p' = np/(n - p)$, or $q = p(n - 1)/(n - p)$, we may use (3.3) to obtain

$$\left(\int_{\Omega_\rho(x)} |u - \bar{u}_{x,\rho}|^{(q-1)p'} dy \right)^{1/(p'(q-1))} \leq C \frac{r^{n-1}}{(r - \rho)^{n-1}} \left(\int_{\Omega_r(x)} |\nabla u|^p dy \right)^{1/p}.$$

If we have the relation $qn/(n - 1) = np/(n - p)$, then the estimate (3.3) will give us

$$\left(\int_{\Omega_\rho(x)} |u - \bar{u}_{x,\rho}|^{qn/(n-1)} dy \right)^{1/p} \leq C \frac{r^{n-1}}{(r - \rho)^{n-1}} \left(\int_{\Omega_r(x)} |\nabla u|^p \right)^{(n-1)/nq}.$$

Simplifying gives that q and p are related by $1/p = 1/2 + 1/(2q)$ if $n = 2$ or $1/p = 1/n + (n-1)/(nq)$ in general, which gives (3.4).

B   The regularity problem in $H^{1,p}(\partial\Omega)$.

The goal of this appendix is to treat the L^p-regularity problem for the Lamé operator L when the data lies in the Hardy-Sobolev space $H^{1,p}(\partial\Omega)$ (see section 2.2 for the definition of this space). By the L^p-regularity problem, we mean the following boundary value problem

$$\begin{cases} Lu = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \\ (\nabla u)^* \in L^p(\partial\Omega). \end{cases} \qquad (\text{B.1})$$

The boundary values are taken in the sense of non-tangential limits. The argument we give is adapted from an argument of Pipher and Verchota [27] used to study boundary value problems for the bi-harmonic operator in three dimensions and was subsequently used to study the traction problem and regularity problem for the Lamé system by Dahlberg and Kenig [10] for $p > 1$. Our main result is the following theorem which treats the L^p-regularity problem in two and three dimensions and $p \leq 1$.

Theorem B.2 *Let $\Omega \subset \mathbf{R}^n$, with $n = 2$ or 3, be a Lipschitz domain and suppose that L is the Lamé operator with coefficients satisfying (2.6).*

There exists $p_1 < 1$ so that regularity problem (B.1) has a solution for $p_1 < p \leq 1$. More precisely we have:

If f is in $H^{1,p}(\partial\Omega)$, $p_1 < p \leq 1$, then there exists a solution of the L^p-regularity problem which satisfies

$$\|\frac{\partial u}{\partial \rho}\|_{H^p(\partial\Omega)} + \|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|f\|_{H^{1,p}(\partial\Omega)}.$$

The constant in this estimate depends on M and the global character of Ω.

Furthermore, if $p > p_1$ and u is a solution of the L^p-regularity problem, (B.1), with $f = 0$, then $u = 0$.

We recall several facts that will be useful in our proof. A real-variable estimate of Dindoš and Mitrea [13, Lemma 6.1] tells us that there is a constant C so that

$$\left(\int_\Omega |u|^p \, dy \right)^{1/p} \leq C \left(\int_{\partial\Omega} (u^*)^{p(n-1)/n} \, d\sigma \right)^{n/p(n-1)}, \qquad p > 0. \qquad (\text{B.3})$$

In work from 1988, Dahlberg, Kenig, and Verchota [11] treated the Dirichlet problem for the Lamé system (also see work of Mayboroda and Mitrea if $n = 2$

[21]). From this work and a real-variable argument we can show that there exists $\epsilon > 0$ so that the L^p-Dirichlet problem

$$\begin{cases} Lu = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \\ u^* \in L^p(\partial\Omega) \end{cases} \tag{B.4}$$

has a unique solution for p in the range $(2 - \epsilon, 2 + \epsilon)$. This solution satisfies the estimate $\|u^*\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}$.

The following technical result is the main step in our argument. The proof follows ideas of Pipher and Verchota and especially Dahlberg and Kenig [10, Lemma 1.6]. This Lemma will use the sets Σ_k as defined in section 4.

Lemma B.5 *Suppose Ω is a Lipschitz domain in $\mathbf{R}^n$, $n = 2, 3$, and L is the Lamé operator with coefficients satisfying the conditions (2.6). If A is a 1-atom supported in a boundary ball $\Delta_r(x)$, then there exists $\gamma > 0$ so that*

$$\left(\int_{\Sigma_k} |\nabla u|^2 \, d\sigma \right)^{1/2} \le C 2^{-\gamma k} (2^k r)^{-(n-1)/2}, \qquad k = 0, 1, 2, \ldots. \tag{B.6}$$

and for $1 \ge p > 1/(1 + \gamma)$,

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \le C\sigma(\Delta_r(x))^{\frac{1}{p} - 1}.$$

Proof. We let A be a 1-atom supported in a boundary ball $\Delta_r(x)$. From the work of Dahlberg, Kenig, and Verchota [11, Theorem 3.7] (see also Mayboroda and Mitrea [21] for results in two dimensions), we have a solution of the regularity problem (B.1) with $(\nabla u)^* \in L^2(\partial\Omega)$ that satisfies the estimate

$$\|(\nabla u)^*\|_{L^2(\partial\Omega)} \le C\|\nabla_t A\|_{L^2(\Delta_r(x))} \le C\sigma(\Delta_r(x))^{-1/2}.$$

This implies the estimate (B.6) with a constant that depends on k. Thus, it suffices to prove (B.6) for $k \ge k_0$. As in the proof of Lemma 4.10 we choose k_0 so that $\text{dist}(\Delta_r(x), \Sigma_{k_0}) \approx r$.

To establish (B.6) for $k \ge k_0$, fix $R > r$ and suppose that $\Delta_{8R}(y)$ is a boundary ball with $\Delta_{8R}(y) \cap \Delta_r(x) = \emptyset$. We use the results Lemma 4.4 and (4.12) to obtain

$$\int_{\Delta_R(y)} |\nabla u|^2 \, d\sigma \le \frac{C}{R} \int_{\Omega_{2R}(y)} |\nabla u|^2 \, dz \le \frac{C}{R^3} \int_{\Omega_{4R}(y)} |u|^2 \, dz. \tag{B.7}$$

We let $E(y, R) = \{z \in \partial\Omega : \Gamma(z) \cap \Omega_{4R}(y) \ne \emptyset\}$ and observe that $E(y, R) \subset \{z : |y - z| \le CR\}$ where $C = C(M, \alpha)$. We fix an exponent t in $(2 - \epsilon, 2)$ such that we can solve the L^t-Dirichlet problem. The estimate of Dindoš and Mitrea (B.3), Hölder's inequality and the solvability of the L^t-Dirichlet problem imply

that we have

$$\frac{1}{R^3} \int_{\Omega_{4R}(y)} |u|^2 \, dz \leq \frac{C}{R^3} \left(\int_{E(y,R)} (u^*)^{2(n-1)/n} \, d\sigma \right)^{(n-1)/n}$$

$$\leq C R^{n-3-(n-1)2/t} \|u^*\|^2_{L^t(\partial\Omega)} \leq C \|A\|^2_{L^t(\Delta_r(x))} R^{n-3-(n-1)2/t}$$

$$\leq C r^{(n-1)2/t+4-2n} R^{n-3-(n-1)2/t}.$$

$$(B.8)$$

We have used the properties of the atom A to obtain that $|A| \leq C/r^{n-2}$ in the last line. If we combine (B.7) and (B.8) we obtain

$$\int_{\Delta_R(y)} |\nabla u|^2 \, d\sigma \leq C \left(\frac{r}{R} \right)^{3-n-(n-1)(2/t-1)} \sigma(\Delta_R(y))^{-1}. \qquad (B.9)$$

We let $\gamma = \frac{1}{2}((n-3) + (n-1)(\frac{2}{t}-1))$. Since $t < 2$, γ will be positive if $n \leq 3$. The estimate (B.6) follows from (B.9) by an elementary covering argument.

The second estimate follows using a representation formula for the gradient of a solution u as in the proof of Theorem 4.1. Note that it is much easier to justify the integration by parts in this case as we have $(\nabla u)^* \in L^2(\partial\Omega)$. ∎

Now we are ready to give the proof of the main result of this appendix.

Proof of Theorem B.2. Let $f = \sum_j \lambda_j A_j$ with each A_j a 1-atom supported in $\Delta_j = \Delta_{r_j}(x_j)$ and $\sum_j \lambda_j^p \sigma(\Delta_j)^{1-p} < \infty$. We let v_j be the solution of (B.1) with $f = A_j$ as given in the work of Dahlberg, Kenig, and Verchota [11] and consider the sum $u = \sum_j \lambda_j v_j$. Using the estimate (B.3) and the Poincaré inequality, we have that as long as $p > (n-1)/n$,

$$\|v_j\|_{L^{pn/(n-1)}(\Omega)} \leq C \|\nabla v_j\|_{L^{pn/(n-1)}(\Omega)} \leq C \|(\nabla v_j)^*\|_{L^p(\partial\Omega)} \leq C \sigma(\Delta_j)^{\frac{1}{p}-1}.$$

The elementary inequality $(\sum a_j)^p \leq \sum a_j^p$, valid if $a_j \geq 0$ and $p < 1$, implies that for $M < M'$, we have

$$\| \sum_{j=M}^{M'} \lambda_j v_j \|_{L^{pn/(n-1)}(\Omega)} \leq \left(\sum_{j=M}^{M'} \lambda_j^p \sigma(\Delta_j)^{1-p} \right)^{1/p}$$

which implies that the series defining u converges at least in $L^{pn/(n-1)}(\Omega)$. The estimate for $\|(\nabla v_j)^*\|_{L^p(\partial\Omega)}$ in Lemma B.5 implies that

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C \left(\sum_j \lambda_j^p \sigma(\Delta_j)^{1-p} \right)^{1/p}, \qquad p > 1/(1+\gamma)$$

and that ∇u has non-tangential limits a.e. in $\partial\Omega$. We may follow the argument in Theorem 6.1 to show that $\partial u/\partial\rho$ lies in $H^p(\partial\Omega)$. This completes the proof of

existence of solutions to the L^p-regularity problem for $p > \max\{(n-1)/n, 1/(1+\gamma)\}$.

We establish the uniqueness of solutions to the L^p-regularity problem. Suppose u is a solution of (B.1) with $u = 0$ a.e. on $\partial\Omega$. From Lemma 5.1 or the result in Mitrea and Wright [24, Lemma 11.9] for $n = 3$, we obtain that $u^* \in L^q(\partial\Omega)$, $1/q = 1/p - 1/(n-1)$. Thus, if $q > 2 - \epsilon$ with $2 - \epsilon$ the lower bound for uniqueness in the Dirichlet problem from the work of Dahlberg, Kenig, and Verchota [11], we may conclude that $u = 0$. In two dimensions, this gives uniqueness if $p > 1 - 1/(3 - \epsilon)$ and in three dimensions we obtain uniqueness if $p > 1 - \epsilon/(4 - \epsilon)$. ∎

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